1. If $n$ and $m$ are integers and $\hat{a}$ and $\hat{a}^\dagger$ are the boson annihilation and creation operators, respectively, show that

\begin{align*}
\bullet\quad & \hat{a}^m |n\rangle = \sqrt{n!/(n-m)!} |n-m\rangle \\
\bullet\quad & (\hat{a}^\dagger)^m |n\rangle = \frac{(\hat{a}^\dagger)^{n+m}}{\sqrt{n!}} |0\rangle = \frac{(n+m)!}{n!} |n+m\rangle \\
\bullet\quad & \exp(\ii t\hat{a}) |k\rangle = \sum_{l=0}^{k} \frac{(\ii t)^l}{l!} \sqrt{\frac{k!}{(k-l)!}} |k-l\rangle \quad \text{for} \quad l \leq k. \\
\bullet\quad & \langle m \mid \exp(\ii t\hat{a}^\dagger) = \sum_{n=0}^{m} \frac{(\ii t)^n}{n!} \sqrt{\frac{m!}{(m-n)!}} \langle m-n \mid \quad \text{for} \quad n \leq m. \\
\bullet\quad & [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger^m] = m\hat{a}^\dagger m \\
\bullet\quad & [\hat{a}^\dagger \hat{a}, \hat{a}^m] = -m\hat{a}^m \\
\bullet\quad & \exp(\ii \xi \hat{a}^\dagger \hat{a}) |0\rangle = f(\hat{a}^\dagger + \xi) |0\rangle \\
\bullet\quad & \exp(\ii \xi \hat{a}^\dagger \hat{a}) f(\hat{a}^\dagger) |0\rangle = f(\hat{a}^\dagger \exp(-\xi)) |0\rangle \\
\bullet\quad & \exp(\ii \xi \hat{a}^\dagger) \hat{a}^m = (\hat{a} - \xi)^m \exp(\ii \hat{a}^\dagger) \\
\bullet\quad & \exp(\ii \xi \hat{a}) \hat{a}^\dagger m = (\hat{a}^\dagger + \xi)^m \exp(\ii \hat{a})
\end{align*}

2. The evolution of a quantum particle with mass $m = 1$ in free space is governed by the Schrödinger equation

\[ i\frac{\partial}{\partial t} |\Psi(x, t)\rangle = -\frac{1}{2} \frac{\partial^2}{\partial x^2} |\Psi(x, t)\rangle. \]

Thus, by defining the momentum operator $\hat{p} = -i\frac{\partial}{\partial x}$, such that the commutator $[\hat{x}, \hat{p}] = i$, it is clear that the formal solution is

\[ |\Psi(x, t)\rangle = \exp \left( \frac{\ii \hat{p}^2 t}{2} \right) |\Psi(x, 0)\rangle. \]

Assume that the initial wavefunction is given as the product of two functions $|\Psi(x, 0)\rangle = g(x)f(x)$. Develop the expression for $|\Psi(x, t)\rangle$ utilizing the identities given in exercise
number 10 in the first set of exercises. Start by expanding the initial wavefunction $|\Psi(x,0)\rangle = g(x)f(s)$ in plane waves, namely

$$|\Psi(x,t)\rangle = \exp \left( \frac{itp^2}{2} \right) \left[ \int_{-\infty}^{\infty} G(u) \exp (ixu) \, du \right] \left[ \int_{-\infty}^{\infty} F(v) \exp (ixv) \, dv \right].$$

Write your result for the particular case when $g(x) = \exp \left( -\frac{x^2}{2\sigma^2} \right)$. In practice this Gaussian function is known as apodization function and it serves to create finite wavefunctions.