Fluctuation-induced Phenomena: Problem Set 6

1) Driven harmonic oscillator. Consider the differential equation for the driven harmonic oscillator:
\[ \ddot{x}(t) + \gamma \dot{x}(t) + \omega_0^2 x(t) = F(t). \] (1)

For a general driving force \( F(t) \), which is significantly nonzero for a finite time interval only, write down an expression for the special solution.

2) Modes and dissipation. Consider a system of coupled harmonic oscillators described by the Hamiltonian
\[ \hat{H} = \frac{\dot{p}^2}{2M} + \frac{1}{2} M \omega_0^2 \hat{x}^2 + \sum_{n=1}^{\infty} \frac{1}{2m_n} \left( \dot{p}_n^2 + m_n^2 \omega_n^2 [\hat{q}_n - \hat{x}]^2 \right). \] (2)

The oscillator with mass \( M \) and frequency \( \omega_0 \) is our system, while the oscillators with mass \( m_n \) and frequency \( \omega_n \) describe the environment.

(a) Show that the previous Hamiltonian leads to the following equation of motions
\[ \ddot{x}(t) + \omega_0^2 \hat{x}(t) = \sum_{n=1}^{\infty} \frac{m_n}{M} \omega_n^2 [\hat{q}_n - \hat{x}], \quad \ddot{\hat{q}}_n(t) + \omega_n^2 \hat{q}_n(t) = \omega_n^2 \hat{x}(t) \] (3)

(b) Go in the Fourier space and show that combining the equations one obtains
\[ \left[ -\omega^2 + \omega_0^2 - \omega^2 \sum_{n=1}^{\infty} \frac{m_n}{M} \frac{\omega_n^2}{\omega_n^2 - \omega^2} \right] \hat{x}(\omega) = 0 \] (4)

(c) If we now consider the case where the frequencies \( \omega_n \) become dense to form a continuum, one can write
\[ \left[ -\omega^2 + \omega_0^2 - \omega^2 \sum_{n=1}^{\infty} \frac{m_n}{M} \frac{\omega_n^2}{\omega_n^2 - \omega^2} \right] \rightarrow -\omega^2 + \omega_0^2 - i\omega \mu(\omega) \]

with \( \mu(\omega) = -i\omega \int_0^{\infty} d\nu \rho(\nu) \frac{m(\nu)}{M} \frac{\nu^2}{\nu^2 - \omega^2} \), (5)

where \( \rho(\nu) \) denotes the density of frequencies enabling us to transform the sum into an integral. Using \( \rho(\nu) \frac{m(\nu)}{M} \nu^2 = \frac{2}{\pi} \gamma \), evaluate the integral in \( \mu(\omega + i\epsilon) \) and consider \( \epsilon \rightarrow 0 \) at the end of the calculation. Discuss the implications of the sign of \( \epsilon \).