Humboldt-Universität zu Berlin
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AG Theoretische Optik & Photonik
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Discrete Quantum Optics

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1. If n and m are integers and $\hat{a}$ and $\hat{a}^\dagger$ are the boson annihilation and creation operators, respectively, show that

- $\hat{a}^m |n\rangle = \sqrt{\frac{n!}{(n-m)!}} |n-m\rangle = \frac{\sqrt{n!}}{(n-m)!} (\hat{a}^\dagger)^{n-m} |0\rangle$
- $(\hat{a}^\dagger)^m |n\rangle = \frac{(\hat{a}^\dagger)^{n+m}}{\sqrt{n!}} |0\rangle = \sqrt{\frac{(n+m)!}{n!}} |n+m\rangle$
- $\exp(\mathrm{i}t\hat{a}) |k\rangle = \sum_{l=0}^{k} \frac{(\mathrm{i}t)^l}{l!} \sqrt{\frac{k!}{(k-l)!}} |k-l\rangle \quad for \quad l \leq k.$
- $\langle m| \exp\left(\mathrm{i}t\hat{a}^\dagger\right) = \sum_{n=0}^{m} \frac{(\mathrm{i}t)^n}{n!} \sqrt{\frac{m!}{(m-n)!}} \langle m-n| \quad for \quad n \leq m.$
- $\left[\hat{a}^\dagger\hat{a}, \hat{a}^{\dagger m}\right] = m\hat{a}^{\dagger m}$
- $\left[\hat{a}^\dagger\hat{a}, \hat{a}^{m}\right] = -m\hat{a}^{m}$
- $\exp(\xi\hat{a})\, f(\hat{a}^\dagger) |0\rangle = f(\hat{a}^\dagger + \xi) |0\rangle$
- $\exp\left(\xi\hat{a}^\dagger\hat{a}\right) f(\hat{a}^\dagger) |0\rangle = f\left(\hat{a}^\dagger \exp(-\xi)\right) |0\rangle$
- $\exp\left(\xi\hat{a}^\dagger\right) \hat{a}^m = (\hat{a} - \xi)^m \exp\left(\xi\hat{a}^\dagger\right)$
- $\exp(\xi\hat{a})\, \hat{a}^{\dagger m} = (\hat{a}^\dagger + \xi)^m \exp(\xi\hat{a})$

2. The evolution of a quantum particle with mass $m = 1$ in free space is governed by the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t} |\Psi(x,t)\rangle = -\frac{1}{2}\frac{\partial^2}{\partial x^2} |\Psi(x,t)\rangle.$$

Thus, by defining the momentum operator $\hat{p} = -\mathrm{i}\frac{\partial}{\partial x}$, such that the commutator $[\hat{x}, \hat{p}] = \mathrm{i}$, it is clear that the formal solution is

$$|\Psi(x,t)\rangle = \exp\left(\frac{\mathrm{i}t\hat{p}^2}{2}\right) |\Psi(x,0)\rangle.$$

Assume that the initial wavefunction is given as the product of two functions $|\Psi(x,0)\rangle = g(x)f(x)$. Develop the expression for $|\Psi(x,t)\rangle$ utilizing the identities given in exercise number 10 in the first set of exercises. Start by expanding the initial wavefunction $|\Psi(x,0)\rangle = g(x)f(s)$ in plane waves, namely

$$|\Psi(x,t)\rangle = \exp\left(\frac{\mathrm{i}t\hat{p}^2}{2}\right)\left[\int_{-\infty}^{\infty} G(u)\exp\left(\mathrm{i}xu\right)du\right]\left[\int_{-\infty}^{\infty} F(v)\exp\left(\mathrm{i}xv\right)dv\right].$$

Write your result for the particular case when $g(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right)$. In practice this Gaussian function is known as apodization function and it serves to create finite wavefunctions.