Discrete Quantum Optics

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1. Consider the Hamiltonian $\hat{H} = \begin{pmatrix} \beta_1 & \kappa \\ \kappa & \beta_2 \end{pmatrix}$, use mathematica, matlab, maple, etc, to obtain the eigenvalues and eigenvectors. Compute numerically the evolution operator $\hat{U}(t) = \exp\left(-\mathrm{i}t\hat{H}\right)$ using the spectral decomposition

$$\sum_{n=1}^{2} \exp\left(-i\lambda_n t\right) |\varphi_n\rangle \langle\varphi_n| . \tag{1}$$

Consider the initial state $|\Psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and describe the behavior of $|\Psi(t)\rangle$ for the cases $\beta_1 >> \beta_2$ and $\beta_1 = \beta_2$.

Compute the coefficients $C_n(t)$ for the expression $|\Psi(t)\rangle = \sum_n C_n(t) |\varphi_n\rangle$ when $|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (|\varphi_1\rangle + |\varphi_2\rangle)$. Explain what is the physical significance of $C_n(t)$?

2. Write down the differential equation

$$i\frac{da_n}{dt} + E_0 a_n + C\left(a_{n+1} + a_{n-1}\right) = 0, \tag{2}$$

in matrix form for $n \in [1, 50]$, and use the spectral decomposition to compute numerically the state vector $|\Psi(t)\rangle = (a_1(t), ..., a_{25}(t), ..., a_{50}(t))^T$ when the initial condition is $|\Psi(0)\rangle = (0_1, ..., 1_{25}, ...0_{50})^T$, where T means transpose. For this exercise consider $\kappa = \beta = 1$ and $t \in [0, 5]$.

Verify that $a_n(t) = A_0(i)^n \exp\left(iE_0 t\right) J_n\left(2Ct\right)$, where A_0 is a constant and $J_n(x)$ is a Bessel function of order n, satisfies the above differential equation, Eq. (2). Assuming the above initial condition integrate numerically Eq. (2) and compare it with the analytical solution.

Assume that the system is formed by an infinite number of sites. Considering the Ansatz $a_n = A\exp\left(-ik_x nQ\right)\exp\left(-iEt\right)$, show that the energy E is given by

$$E = E_0 + 2C\cos\left(k_x Q\right). \tag{3}$$

Note, $k_x Q$ is commonly know as the Bloch momentum and Q is the spacing between the sites. Plugging the expression for E into a_n describe the physical significance of the solution.