Discrete Quantum Optics

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1. Show that the eigenvectors of a linear operator $\hat{A}$ are linearly independent.

2. Show that $\left(\hat{A}+\hat{B}\right)^{\dagger}=\hat{A}^{\dagger}+\hat{B}^{\dagger}$, and $\left(\hat{A}\hat{B}\right)^{\dagger}=\hat{B}^{\dagger}\hat{A}^{\dagger}$.

3. Show that a projector operator $\hat{P}$ is Hermitian and can be inverted if and only if $\hat{P}=I$, where I is the identity operator.

4. Show that the standard Pauli matrices σ_x, σ_y, and σ_z are Hermitian and unitary.

5. Let be $|+\rangle=\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $|-\rangle=\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$, write down the Pauli matrices in the basis $\{|+\rangle,|-\rangle\}$.

6. Show that if the operators $\hat{A}$ and $\hat{B}$ are Hermitian, then $\mathrm{i}\left[\hat{A},\hat{B}\right]$ is also Hermitian.

7. Show that any unitary operator $\hat{U}$ can be written as $\hat{U}=\exp\left(\mathrm{i}\hat{A}\right)$, where $\hat{A}$ is a Hermitian operator.

8. Consider the Hamiltonian $\hat{H}=\left(\begin{smallmatrix}\beta&\kappa\\\kappa&\beta\end{smallmatrix}\right)$, compute the evolution operator $\hat{U}(t)=\exp\left(-\mathrm{i}t\hat{H}\right)$ using (i) the spectral decomposition and (ii) expanding the exponential in Taylor series.

9. Given the operator momentum operator $\hat{p}=-\mathrm{i}\frac{\partial}{\partial x}$, show that the commutator $[\hat{x},\hat{p}]=\mathrm{i}$.

10. Show the identities:

- $\exp\left(\frac{\mathrm{i}t\hat{p}^2}{2}\right)h(\hat{x})\exp\left(\frac{-\mathrm{i}t\hat{p}^2}{2}\right)=h(\hat{x}+t\hat{p})$
- $\exp\left(\frac{\mathrm{i}u(\hat{x}+t\hat{p})}{2}\right)=\exp\left(\frac{\mathrm{i}tu^2}{2}\right)\exp\left(\mathrm{i}u\hat{x}\right)\exp\left(\mathrm{i}tu\hat{p}\right)$
- $\exp\left(\frac{\mathrm{i}t\hat{p}^2}{2}\right)\exp\left(\mathrm{i}v\hat{x}\right)=\exp\left(\frac{\mathrm{i}tv^2}{2}\right)\exp\left(\mathrm{i}v\hat{x}\right)$
- $\exp\left(\mathrm{i}tu\hat{p}\right)\exp\left(\mathrm{i}v\hat{x}\right)=\exp\left(\mathrm{i}tuv\right)\exp\left(\mathrm{i}v\hat{x}\right)$
- $\exp\left(\frac{\mathrm{i}t\hat{p}^2}{2}\right)\exp\left(\frac{-\mathrm{i}t\hat{x}^2}{2\sigma^2}\right)\exp\left(\frac{-\mathrm{i}t\hat{p}^2}{2}\right)=\exp\left[\frac{-\left(\hat{x}^2+(\hat{x}\hat{p}+\hat{p}\hat{x})+t^2\hat{p}^2\right)}{2\sigma^2}\right]$

11. Consider the spectral decomposition of the Fourier operator $\hat{F}_\alpha=\sum_{n=0}^{\infty}|\varphi_n\rangle\langle\varphi_n|\exp(\mathrm{i}n\alpha)$, where $|\varphi_n\rangle$ represents a normalized Hermite-Gauss polynomials of order n.
Using matlab or mathematica, maple, etc, to apply such a Fourier operator with $\alpha=\frac{\pi}{2},\frac{\pi}{4}$ to a rect function defined as $\Pi(x)=\begin{cases}0 & \text{if} \quad |x|>1\\ 1 & \text{if} \quad |x|=1\\ 0 & \text{if} \quad |x|<1\end{cases}$.
