HUMBOLDT–UNIVERSITÄT ZU BERLIN
INSTITUT FÜR PHYSIK

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Fluctuation-induced Phenomena: Problem Set 7

1) Free space Green tensor (5 Points). In free space, the Green tensor is the solution to the equation

$$\nabla \times \nabla \times \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) - k_0^2 \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}'), \tag{1}$$

where $k_0 = \omega/\mathrm{c}$ is the ratio of the angular frequency to the speed of light.

(a) Take the divergence of both sides of Eq. (1) and show that

$$\nabla \cdot \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = -\frac{1}{k_0^2} \nabla \delta(\mathbf{r} - \mathbf{r}'). \tag{2}$$

(b) Use the vectorial identity $\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}$ to rewrite Eq. (1) as

$$\nabla^2 \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) + k_0^2 \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = -\left[1 + \frac{1}{k_0^2} \nabla\nabla\right] \delta(\mathbf{r} - \mathbf{r}'). \tag{3}$$

(c) If we now define

$$\mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = -\left[1 + \frac{1}{k_0^2} \nabla\nabla\right] g_0(\mathbf{r}, \mathbf{r}', \omega), \tag{4}$$

show that $g_0(\mathbf{r}, \mathbf{r}', \omega)$ is the (scalar) Green function of the Helmholtz equation:

$$\nabla^2 g_0(\mathbf{r}, \mathbf{r}', \omega) + k_0^2 g_0(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r}, \mathbf{r}', \omega). \tag{5}$$

2) The Green function of the Helmholtz equation (5 Points).

(a) By transforming to the spatial Fourier space, show that the Green tensor of the Helmholtz equation is given by the integral

$$g_0(\mathbf{r}, \mathbf{r}', \omega) = -\frac{1}{(2\pi)^3} \int \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{|\mathbf{k}|^2 - k_0^2} \mathrm{d}^3\mathbf{k}. \tag{6}$$

(b) Calculate the integral considering the boundary condition that for a fixed $\mathbf{r}'$, $g_0(\mathbf{r}, \mathbf{r}', \omega)$ must be an outgoing wave (traveling away from $\mathbf{r}'$) in the limit $|\mathbf{r} - \mathbf{r}'| \to \infty$. This condition is known as the Sommerfeld radiation condition.
Hint: You can control the boundary condition by a small displacement of the pole of the integrand.

— 10 Points —

— To be handed in prior to the tutorials on 19.06.2016 —