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Discrete Quantum Optics

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1. Consider a quantum system initially prepared in a number state $|\Psi(0)\rangle = |k\rangle$, governed by the hamiltonian $H = \left(\hat{a}^\dagger + \hat{a}\right)$, that is, a system with a Hamiltonian proportional to the $\hat{x}$ operator, $\hat{x} \propto \left(\hat{a}^\dagger + \hat{a}\right)$.

In order to solve the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = H\left|\Psi(t)\right\rangle, \tag{1}$$

we start by expanding the state vector $|\Psi(t)\rangle$ in terms of number states

$$\left|\Psi(t)\right\rangle = \sum_{m=0}^{\infty}\left|m\right\rangle\left\langle m|\Psi(t)\right\rangle = \sum_{m=0}^{\infty} C_m(t)\left|m\right\rangle, \tag{2}$$

where we have defined $C_m(t) = \langle m|\Psi(t)\rangle = \langle m| U(t) |k\rangle$. Substituting expansion (2) into the Schrödinger equation yields

$$\mathrm{i}\sum_{m=0}^{\infty}\frac{\partial C_m(t)}{\partial t}\left|m\right\rangle = \sum_{m=0}^{\infty} C_m(t)\hat{a}^\dagger\left|m\right\rangle + \sum_{m=0}^{\infty} C_m(t)\hat{a}\left|n\right\rangle. \tag{3}$$

Since $\hat{a}^\dagger|m\rangle = \sqrt{m+1}\,|m+1\rangle$ and $\hat{a}|m\rangle = \sqrt{n}\,|m-1\rangle$ we have

$$\mathrm{i}\sum_{m=0}^{\infty}\frac{\partial C_m(t)}{\partial t}\left|m\right\rangle = \sum_{m=0}^{\infty} C_m(t)\sqrt{m+1}\left|m+1\right\rangle + \sum_{m=0}^{\infty} C_m(t)\sqrt{m}\left|m-1\right\rangle. \tag{4}$$

Multiplying to the left by $\langle n|$ we obtain

$$\mathrm{i}\sum_{n=0}^{\infty}\frac{\partial C_m(t)}{\partial t}\left\langle n|m\right\rangle = \sum_{m=0}^{\infty} C_m(t)\sqrt{m+1}\left\langle n|m+1\right\rangle + \sum_{m=0}^{\infty} C_m(t)\sqrt{m}\left\langle n|m-1\right\rangle. \tag{5}$$

Using the orthogonality relation $\langle n|m\rangle = \delta_{n,m}$, Eq. (5) reduces to

$$\mathrm{i}\frac{\partial C_m(t)}{\partial t} = \sqrt{m}C_{m-1}(t) + \sqrt{m+1}C_{m+1}(t). \tag{6}$$

Or in matrix form

$$\mathrm{i}\frac{\partial}{\partial t}\begin{bmatrix} C_0(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{1} & 0 & \dots & 0 \\ \sqrt{1} & 0 & \sqrt{2} & \dots & 0 \\ 0 & \sqrt{2} & 0 & \dots & \vdots \\ 0 & 0 & \sqrt{3} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 \end{bmatrix} \begin{bmatrix} C_0(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \\ \vdots \end{bmatrix}.$$

Instead of solving the set of coupled differential equation (6), it is easier to compute $C_m(t)$ directly from the formal solution of the Schrödinger equation

$$|\Psi(t)\rangle = \exp\left(-\mathrm{i}t\left(\hat{a}^\dagger + \hat{a}\right)\right)|k\rangle\,. \tag{7}$$

Since $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, we can use the Baker-Campbell-Hausdorff formula to split the exponential operator

$$\exp\left(-\mathrm{i}t\left(\hat{a}^\dagger + \hat{a}\right)\right) = \exp\left(-\frac{t^2}{2}\right)\exp\left(-\mathrm{i}t\hat{a}^\dagger\right)\exp\left(-\mathrm{i}t\hat{a}\right). \tag{8}$$

Therefore, we have

$$|\Psi(t)\rangle = \exp\left(-\frac{t^2}{2}\right)\exp\left(-\mathrm{i}t\hat{a}^\dagger\right)\exp\left(-\mathrm{i}t\hat{a}\right)|k\rangle\,. \tag{9}$$

Since $C_m(t) = \langle m|\Psi(t)\rangle = \langle m|\,U(t)\,|k\rangle$, we have to multiply Eq. (9) to the left by $\langle m|$

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right)\langle m|\exp\left(-\mathrm{i}t\hat{a}^\dagger\right)\exp\left(-\mathrm{i}t\hat{a}\right)|k\rangle\,. \tag{10}$$

We now use the Taylor expansions

$$\exp\left(-\mathrm{i}t\hat{a}\right)|k\rangle = \sum_{l=0}^{\infty}\frac{(-\mathrm{i}t)^l}{l!}(\hat{a})^l\,|k\rangle = \sum_{l=0}^{k}\frac{(-\mathrm{i}t)^l}{l!}\sqrt{\frac{k!}{(k-l)!}}\,|k-l\rangle\,, \tag{11}$$

and

$$\langle m|\exp\left(-\mathrm{i}t\hat{a}^\dagger\right) = \sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^n}{n!}\langle m|\,(\hat{a}^\dagger)^n = \sum_{n=0}^{m}\frac{(-\mathrm{i}t)^n}{n!}\sqrt{\frac{m!}{(m-n)!}}\,\langle m-n|\,, \tag{12}$$

to write Eq. (10) as

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right)\sum_{n=0}^{m}\sum_{l=0}^{k}\frac{(-\mathrm{i}t)^n(-\mathrm{i}t)^l}{n!\quad l!}\sqrt{\frac{m!\quad k!}{(m-n)!(k-l)!}}\,\langle m-n|k-l\rangle\,. \tag{13}$$

Notice that the orthonormality condition, $\langle m-n|k-l\rangle = \delta_{m-n,k-l}$, allows us to take $n = m+l-k$, and as a result Eq. (13) reduces to

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right)\sum_{l=0}^{k}\frac{(-\mathrm{i}t)^{2l}(-\mathrm{i}t)^{m-k}}{(m+l-k)!\quad l!}\sqrt{\frac{m!\quad k!}{(m-n)!(k-l)!}}. \tag{14}$$

Moreover, taking $m = k + s$ and multiplying by $\sqrt{\frac{(k+s)!}{(k+s)!}}$ we obtain

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right)(-\mathrm{i}t)^s\sqrt{\frac{k!}{(k+s)!}}\sum_{l=0}^{k}\frac{(-1)^l(t^2)^l(k+s)!}{l!(l+s)!(k-l)!}. \tag{15}$$

From this expression we identify the associated Laguerre polynomials

$$L_k^s(t^2) = \sum_{l=0}^{k}\frac{(-1)^l(t^2)^l(k+s)!}{l!(l+s)!(k-l)!}. \tag{16}$$

And Eq. (15) becomes

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right)(-\mathrm{i}t)^s\sqrt{\frac{k!}{(k+s)!}}L_k^s(t^2) \qquad \text{for} \qquad m = k+s. \tag{17}$$

Since this expression is valid for $m = k + s$, we can take $s = m - k$ to obtain

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right)(-\mathrm{i}t)^{m-k}\sqrt{\frac{k!}{m!}}L_k^{m-k}(t^2) \qquad \text{for} \qquad m \geq k. \tag{18}$$

Starting again from Eq. (13), and using the orthonormality condition $\langle m-n|k-l\rangle = \delta_{m-n,k-l}$ we now take $l = n + k - m$ to get rid of the sum in l. Then, taking $m = k - s$ and multiplying by $\sqrt{\frac{k!}{k!}}$ we obtain

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right)(-\mathrm{i}t)^s\sqrt{\frac{(k-s)!}{k!}}L_{k-s}^s(t^2) \qquad \text{for} \qquad m = k-s. \tag{19}$$

Finally, taking $s = k - m$ yields

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right)(-\mathrm{i}t)^{k-m}\sqrt{\frac{m!}{k!}}L_m^{k-m}(t^2) \qquad \text{for} \qquad m \leq k. \tag{20}$$

Collecting both parts of the solution we have the analytical expression for the probability amplitudes $C_m(t)$

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right)\times\begin{cases}(-\mathrm{i}t)^{k-m}\sqrt{\frac{m!}{k!}}L_m^{k-m}(t^2) & \text{for} \qquad m \leq k. \\ (-\mathrm{i}t)^{m-k}\sqrt{\frac{k!}{m!}}L_k^{m-k}(t^2) & \text{for} \qquad m \geq k.\end{cases} \tag{21}$$

This expressions describe how the different states $|m\rangle$ will be populated when the system starts in a number state $|k\rangle$. The next step is to write a Matlab script to solve the system of coupled differential equations (6) using the Runge-Kutta method. In doing so consider $m = 0, ..., 50$ and the initial states $|k\rangle = |0\rangle, |2\rangle, |5\rangle$, compare the analytical and numerical solutions.