HUMBOLDT–UNIVERSITÄT ZU BERLIN
INSTITUT FÜR PHYSIK SoSe 2018

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# Fluctuation-induced Phenomena: Problem Set 2

**1) The electromagnetic field in thermal equilibrium at temperature** $T$ (10 Points).

Given the Hamiltonian of the quantum electromagnetic field

$$\hat{H}_{\mathrm{rad}} = \frac{\epsilon_0}{2}\int_V \mathrm{d}^3\mathbf{r}\left[\hat{\mathbf{E}}^2 + c^2\hat{\mathbf{B}}^2\right] = \sum_\mu \hbar\omega_\mu\left(\hat{a}_\mu^\dagger\hat{a}_\mu + \frac{1}{2}\right), \tag{1}$$

show that the mean value of the energy for the field in thermal equilibrium at the temperature $T$ is given by

$$\langle H_{\mathrm{rad}}\rangle = \sum_\mu \hbar\omega_\mu\left(\frac{1}{e^{\frac{\hbar\omega_\mu}{k_{\mathrm{B}}T}} - 1} + \frac{1}{2}\right). \tag{2}$$

Remind that the mean value of the energy is defined as $\langle H_{\mathrm{rad}}\rangle = \mathrm{tr}\left[\hat{H}_{\mathrm{rad}}\hat{\rho}\right]$ where $\hat{\rho}$ is the Gibbs state

$$\hat{\rho} = \frac{1}{Z}e^{-\frac{\hat{H}_{\mathrm{rad}}}{k_{\mathrm{B}}T}}, \text{ with } Z = \mathrm{tr}\left[e^{-\frac{\hat{H}_{\mathrm{rad}}}{k_{\mathrm{B}}T}}\right],$$

In order to simplify the calculation, justify first that that one can write

$$\langle H_{\mathrm{rad}}\rangle = -\partial_\beta \ln Z, \text{ with } \beta = [k_{\mathrm{B}}T]^{-1}. \tag{3}$$

Using the previous expression for the density matrix, calculate the mean value of the electric, $\langle\hat{\mathbf{E}}\rangle$, and of the magnetic field, $\langle\hat{\mathbf{B}}\rangle$, for the the state of thermal equilibrium. Comment the physics behind these results.

— 10 Points —
— To be handed in prior to the tutorials on 30.04.2018 —