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Fluctuation-induced Phenomena: Problem Set 3

1) Calculation of the Casimir effect in one dimension.
Consider two perfectly conducting plates separated by a distance L, as illustrated in Fig. 1.

Figure 1: Geometry for the one dimensional Casimir calculation: two perfectly conducting plates separated by a distance L.

a) First argue that the zero-point energy of the e.m. field between the plates is given as

$$\mathcal{E}_{\text{disc}} = \frac{\pi\hbar c}{L}\sum_{n=1}^{\infty} n. \tag{1}$$

Hint: Find the entire spectrum of modes by solving the wave equation with suitable boundary conditions. Then use the postulate that each mode carries a zero point contribution to the energy of $\hbar\omega/2$.

b) Next, we consider the case where there are no mirrors (or, equivalently, the limit where L becomes large). In this limit, show that the zero-point energy in the same length L can be written as

$$\mathcal{E}_{\text{cont}} = \frac{\pi\hbar c}{L}\int_0^{\infty} n\mathrm{d}n. \tag{2}$$

c) The change in the zero-point energy is evidently

$$\Delta\mathcal{E} = \frac{\pi\hbar c}{L}\left\{\sum_{n=1}^{\infty} n - \int_0^{\infty} n\mathrm{d}n\right\}. \tag{3}$$

Following Casimir, we can evaluate eq. (3) by use of the Euler-Maclaurin formula

$$\sum_{n=a}^{b} f_n - \int_a^b f(n)\mathrm{d}n = -\frac{1}{2}\left(f(b)+f(a)\right) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right), \tag{4}$$

where B_k are the Bernoulli numbers. To this end, we can use a regularization of the integrand by noting that

$$\Delta\mathcal{E} = \frac{\pi\hbar c}{L}\lim_{\epsilon\to 0}\left\{\sum_{n=1}^{\infty} n\mathrm{e}^{-\epsilon n} - \int_0^\infty n\mathrm{e}^{-\epsilon n}\mathrm{d}n\right\}. \tag{5}$$

With Eqs. (4) and (5), calculate the Casimir force as minus the gradient of the (renormalized) energy $\Delta\Delta\mathcal{E}(L)$.

d) More than two centuries ago Niels Henrik Abel and Giovanni Antonio Amedeo Plana discovered independently the following summation formula

$$\sum_{n=0}^{\infty} f(n) = \int_0^\infty \mathrm{d}x\, f(x) + \frac{f(0)}{2} + \mathrm{i}\int_0^\infty \mathrm{d}t\, \frac{f(\mathrm{i}t) - f(-\mathrm{i}t)}{e^{2\pi t} - 1}. \tag{6}$$

Use Eq: (6) to calculate the Casimir force.