Fluctuation-induced Phenomena: Problem Set 3

1) Calculation of the Casimir effect in one dimension.
   Consider two perfectly conducting plates separated by a distance \( L \), as illustrated in Fig. 1.

   Figure 1: Geometry for the one dimensional Casimir calculation: two perfectly conducting plates separated by a distance \( L \).

   a) First argue that the zero-point energy of the e.m. field between the plates is given as

   \[
   E_{\text{disc}} = \frac{\pi \hbar c}{L} \sum_{n=1}^{\infty} n. 
   \]

   \( H \text{int: Find the entire spectrum of modes by solving the wave equation with suitable boundary conditions. Then use the postulate that each mode carries a zero point contribution to the energy of } \hbar \omega/2. \)

   b) Next, we consider the case where there are no mirrors (or, equivalently, the limit where \( L \) becomes large). In this limit, show that the zero-point energy in the same length \( L \) can be written as

   \[
   E_{\text{cont}} = \frac{\pi \hbar c}{L} \int_{0}^{\infty} ndn. 
   \]

   c) The change in the zero-point energy is evidently

   \[
   \Delta E = \frac{\pi \hbar c}{L} \left\{ \sum_{n=1}^{\infty} n - \int_{0}^{\infty} ndn \right\}. 
   \]
Following Casimir, we can evaluate eq. (3) by use of the Euler-Maclaurin formula

\[ \sum_{n=a}^{b} f(n)dn = -\frac{1}{2} (f(b) + f(a)) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (f^{(2k-1)}(b) - f^{(2k-1)}(a)), \]  

(4)

where \( B_k \) are the Bernoulli numbers. To this end, we can use a regularization of the integrand by noting that

\[ \Delta \mathcal{E} = \frac{\pi \hbar c}{L} \lim_{\epsilon \to 0} \left\{ \sum_{n=1}^{\infty} ne^{-\epsilon n} - \int_0^{\infty} ne^{-\epsilon n} \, dn \right\} . \]  

(5)

With Eqs. (4) and (5), calculate the Casimir force as minus the gradient of the (renormalized) energy \( \Delta \Delta \mathcal{E}(L) \).

d) More than two centuries ago Niels Henrik Abel and Giovanni Antonio Amedeo Plana discovered independently the following summation formula

\[ \sum_{n=0}^{\infty} f(n) = \int_{0}^{\infty} dx \, f(x) + \frac{f(0)}{2} + i \int_{0}^{\infty} dt \, \frac{f(it) - f(-it)}{e^{2\pi t} - 1} . \]  

(6)

Use Eq. (6) to calculate the Casimir force.