1) The fluctuation-dissipation theorem. The FDT can be demonstrated using the analytic properties within a stripe of the complex $\tau$-plane of the correlation functions

$$
C_{AB}(\tau) = \langle \hat{A}_0(\tau) \hat{B}_0 \rangle = \text{tr} \left[ \hat{A}_0(\tau) \hat{B}_0 \hat{\rho}_0 \right] \quad \text{and}
$$

$$
C_{BA}(-\tau) = \langle \hat{B}_0(-\tau) \hat{A}_0 \rangle = \text{tr} \left[ \hat{B}_0(-\tau) \hat{A}_0 \hat{\rho}_0 \right],
$$

where $\hat{\rho}_0 = Z^{-1} e^{-\beta \hat{H}_0}$ is the thermal state ($\beta = [k_B T]^{-1}$ and $Z = \text{tr}[e^{-\beta \hat{H}_0}]$) and

$$
\hat{A}_0(t) = e^{i \hat{H}_0 t} \hat{A} e^{-i \hat{H}_0 t}, \quad \hat{B}_0(t) = e^{i \hat{H}_0 t} \hat{B} e^{-i \hat{H}_0 t}.
$$

A more direct demonstration relies on the expansion of the correlation functions over the basis formed by the eigenvectors of the (unperturbed) Hamiltonian $\hat{H}_0 |E\rangle = E |E\rangle$ and on the definition of the density of states $\rho(E)$ which allows us to write the closure relation

$$
1 = \int dE \rho(E) |E\rangle \langle E|.
$$

(1)

Using such an approach, show that

$$
\int_{-\infty}^{\infty} d\tau e^{i \omega \tau} C_{BA}(-\tau) = e^{-\beta \hbar \omega} \int_{-\infty}^{\infty} d\tau e^{i \omega \tau} C_{AB}(\tau).
$$

Hint: Carry out both Fourier transforms and show that they are indeed equal.