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# Fluctuation-induced Phenomena: Problem Set 7

**1) Free space Green tensor**. In free space, the Green tensor is the solution to the equation

$$\nabla \times \nabla \times \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) - k_0^2 \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}'), \tag{1}$$

where $k_0 = \omega/\mathrm{c}$ is the ratio of the angular frequency to the speed of light.

(a) Take the divergence of both sides of Eq. (1) and show that

$$\nabla \cdot \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = -\frac{1}{k_0^2} \nabla \delta(\mathbf{r} - \mathbf{r}'). \tag{2}$$

(b) Use the vectorial identity $\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}$ to rewrite Eq. (1) as

$$\nabla^2 \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) + k_0^2 \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = -\left[1 + \frac{1}{k_0^2} \nabla\nabla\right] \delta(\mathbf{r} - \mathbf{r}'). \tag{3}$$

(c) If we now define

$$\mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = -\left[1 + \frac{1}{k_0^2} \nabla\nabla\right] g_0(\mathbf{r}, \mathbf{r}', \omega), \tag{4}$$

show that $g_0(\mathbf{r}, \mathbf{r}', \omega)$ is the (scalar) Green function of the Helmholtz equation:

$$\nabla^2 g_0(\mathbf{r}, \mathbf{r}', \omega) + k_0^2 g_0(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r}, \mathbf{r}', \omega). \tag{5}$$

**2) The Green function of the Helmholtz equation**.

(a) By transforming to the spatial Fourier space, show that the Green tensor of the Helmholtz equation is given by the integral

$$g_0(\mathbf{r}, \mathbf{r}', \omega) = -\frac{1}{(2\pi)^3} \int \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{|\mathbf{k}|^2 - k_0^2} \mathrm{d}^3\mathbf{k}. \tag{6}$$

(b) Calculate the integral considering the boundary condition that for a fixed $\mathbf{r}'$, $g_0(\mathbf{r}, \mathbf{r}', \omega)$ must be an outgoing wave (traveling away from $\mathbf{r}'$) in the limit $|\mathbf{r} - \mathbf{r}'| \to \infty$. This condition is known as the Sommerfeld radiation condition.
*Hint: You can control the boundary condition by a small displacement of the pole of the integrand.*