Dr. Franceso Intravaia
Humboldt–Universität zu Berlin
Institut für Physik

SoSe 2019

Online on the 08.04.2019
Discussion during the tutorial on the 16.04.2019

Fluctuation-induced Phenomena: Problem Set 1

1) The classical harmonic oscillator. In full analogy with the treatment of the quantum harmonic oscillator, show that for a suitable linear combination α (and its complex conjugate α^*) of p and x, the energy of a *classical* harmonic oscillator can be written as

$$E = \hbar\omega_0\alpha^*\alpha. \tag{1}$$

Derive the equations of motions for α and α^*. What is the minimum energy?

2) Coupled Quantum Harmonic Oscillators
Consider two quantum harmonic oscillators with Hamiltonians

$$\hat{H}_a = \frac{\hat{p}_a^2}{2} + \frac{\omega_a^2\hat{x}_a^2}{2}, \quad \hat{H}_b = \frac{\hat{p}_b^2}{2} + \frac{\omega_b^2\hat{x}_b^2}{2}.$$

Here, we have assumed that the mass is the same for both oscillators and that $m = 1$. The commutation relations are $[\hat{x}_i, \hat{p}_i] = \mathrm{i}\hbar$, while all other operators commute.

(a) Determine the ground state and the associated zero-point energy of the system that consists of the two uncoupled quantum harmonic oscillators, i.e.,

$$\hat{H}_0 = \hat{H}_a + \hat{H}_b.$$

(b) Consider the system of two coupled quantum harmonic oscillators

$$\hat{H}_\mathrm{c} = \hat{H}_\mathrm{a} + \hat{H}_\mathrm{b} + \hat{H}_\mathrm{int} \quad \text{where} \quad \hat{H}_\mathrm{int} = g^2\hat{x}_a\hat{x}_b,$$

where g describes the strength of the coupling. Determine a transformation $\hat{x}_\pm = f(\hat{x}_a, \hat{x}_b)$ and $\hat{p}_\pm = f(\hat{p}_a, \hat{p}_b)$ that provides a decoupling of the Hamiltonian $\hat{H}_\mathrm{c}$ into two independent quantum harmonic oscillators, i.e.,

$$\hat{H}_\mathrm{c} = \frac{\hat{p}_+^2}{2} + \frac{\Omega_+^2\hat{x}_+^2}{2} + \frac{\hat{p}_-^2}{2} + \frac{\Omega_-^2\hat{x}_-^2}{2}.$$

Hint: Consider a general rotation of the coordinates through an angle θ and find the condition on θ for which the system decouples.

(c) Find the commutation relations $[\hat{x}_\pm, \hat{p}_\pm]$ and the zero-point energy associated with $\hat{H}_\mathrm{c}$. Compare this zero-point energy with the corresponding result for the uncoupled oscillators (see (a)).
Hint: Use the relation $\cos^2\theta = 1/(1 + \tan^2\theta)$ to express $\Omega_\pm$ in terms of ω_a, ω_b and g.

(d) What happens when g becomes very large ($g \gg \omega_a,\, \omega_b$)? Give a physical explanation of the corresponding behavior.

3) The ground state force*. Consider the ground state energy of the coupled quantum harmonic oscillators in question 2) above. Assume that the coupling strength depends on the distance L between the oscillators according to $g(L) = \alpha L\, e^{-L/\lambda}$, where α and λ are two constants. Make a plot of the force acting between the two oscillators.