

Discrete Quantum Optics

Dr. Armando Perez-Leija and Prof. Dr. Kurt Busch
M.Sc. Paraschos Varytis

WiSe 2017/18, Set 2
8. November 2017

1. If n and m are integers and \hat{a} and \hat{a}^\dagger are the boson annihilation and creation operators, respectively, show that

- $\hat{a}^m |n\rangle = \sqrt{\frac{n!}{(n-m)!}} |n-m\rangle = \frac{\sqrt{n!}}{(n-m)!} (\hat{a}^\dagger)^{n-m} |0\rangle$
- $(\hat{a}^\dagger)^m |n\rangle = \frac{(\hat{a}^\dagger)^{n+m}}{\sqrt{n!}} |0\rangle = \sqrt{\frac{(n+m)!}{n!}} |n+m\rangle$
- $\exp(it\hat{a}) |k\rangle = \sum_{l=0}^k \frac{(it)^l}{l!} \sqrt{\frac{k!}{(k-l)!}} |k-l\rangle \quad \text{for } l \leq k.$
- $\langle m | \exp(it\hat{a}^\dagger) = \sum_{n=0}^m \frac{(it)^n}{n!} \sqrt{\frac{m!}{(m-n)!}} \langle m-n | \quad \text{for } n \leq m.$
- $[\hat{a}^\dagger \hat{a}, \hat{a}^{\dagger m}] = m \hat{a}^{\dagger m}$
- $[\hat{a}^\dagger \hat{a}, \hat{a}^m] = -m \hat{a}^m$
- $\exp(\xi \hat{a}) f(\hat{a}^\dagger) |0\rangle = f(\hat{a}^\dagger + \xi) |0\rangle$
- $\exp(\xi \hat{a}^\dagger \hat{a}) f(\hat{a}^\dagger) |0\rangle = f(\hat{a}^\dagger \exp(-\xi)) |0\rangle$
- $\exp(\xi \hat{a}^\dagger) \hat{a}^m = (\hat{a} - \xi)^m \exp(\xi \hat{a}^\dagger)$
- $\exp(\xi \hat{a}) \hat{a}^{\dagger m} = (\hat{a}^\dagger + \xi)^m \exp(\xi \hat{a})$

2. The evolution of a quantum particle with mass $m = 1$ in free space is governed by the Schrödinger equation

$$i \frac{\partial}{\partial t} |\Psi(x, t)\rangle = -\frac{1}{2} \frac{\partial^2}{\partial x^2} |\Psi(x, t)\rangle.$$

Thus, by defining the momentum operator $\hat{p} = -i \frac{\partial}{\partial x}$, such that the commutator $[\hat{x}, \hat{p}] = i$, it is clear that the formal solution is

$$|\Psi(x, t)\rangle = \exp\left(\frac{it\hat{p}^2}{2}\right) |\Psi(x, 0)\rangle.$$

Assume that the initial wavefunction is given as the product of two functions $|\Psi(x, 0)\rangle = g(x)f(x)$. Develop the expression for $|\Psi(x, t)\rangle$ utilizing the identities given in exercise

number 10 in the first set of exercises. Start by expanding the initial wavefunction $|\Psi(x, 0)\rangle = g(x)f(s)$ in plane waves, namely

$$|\Psi(x, t)\rangle = \exp\left(\frac{it\hat{p}^2}{2}\right) \left[\int_{-\infty}^{\infty} G(u) \exp(ixu) du \right] \left[\int_{-\infty}^{\infty} F(v) \exp(ixv) dv \right].$$

Write your result for the particular case when $g(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right)$. In practice this Gaussian function is known as apodization function and it serves to create finite wavefunctions.