

Fluctuation-induced Phenomena: Problem Set 1

- 1) **The classical harmonic oscillator** (3 Points). In full analogy with the treatment of the quantum harmonic oscillator, show that for a suitable linear combination of p and x , the energy of a *classical* harmonic oscillator can be written as

$$E = \hbar\omega_0\alpha^*\alpha. \quad (1)$$

Derive the equations of motions for α and α^* . What is the minimum energy?

- 2) **Coupled Quantum Harmonic Oscillators** (7 Points)

Consider two quantum harmonic oscillators with Hamiltonians

$$\hat{H}_a = \frac{\hat{p}_a^2}{2} + \frac{\omega_a^2 \hat{x}_a^2}{2}, \quad \hat{H}_b = \frac{\hat{p}_b^2}{2} + \frac{\omega_b^2 \hat{x}_b^2}{2}.$$

Here, we have assumed that the mass is the same for both oscillators and that $m = 1$. The commutation relations are $[\hat{x}_i, \hat{p}_i] = i\hbar$, while all other operators commute.

- (a) Determine the ground state and the associated zero-point energy of the system that consists of the two uncoupled quantum harmonic oscillators, i.e.,

$$\hat{H}_0 = \hat{H}_a + \hat{H}_b.$$

- (b) Consider the system of two coupled quantum harmonic oscillators

$$\hat{H}_c = \hat{H}_a + \hat{H}_b + \hat{H}_{\text{int}} \quad \text{where} \quad \hat{H}_{\text{int}} = g^2 \hat{x}_a \hat{x}_b,$$

where g describes the strength of the coupling. Determine a transformation $\hat{x}_{\pm} = f(\hat{x}_a, \hat{x}_b)$ and $\hat{p}_{\pm} = f(\hat{p}_a, \hat{p}_b)$ that provides a decoupling of the Hamiltonian \hat{H}_c into two independent quantum harmonic oscillators, i.e.,

$$\hat{H}_c = \frac{\hat{p}_+^2}{2} + \frac{\Omega_+^2 \hat{x}_+^2}{2} + \frac{\hat{p}_-^2}{2} + \frac{\Omega_-^2 \hat{x}_-^2}{2}.$$

Hint: Consider a general rotation of the coordinates through an angle θ and find the condition on θ for which the system decouples.

- (c) Find the commutation relations $[\hat{x}_{\pm}, \hat{p}_{\pm}]$ and the zero-point energy associated with \hat{H}_c . Compare this zero-point energy with the corresponding result for the uncoupled oscillators (see (a)).

Hint: Use the relation $\cos^2 \theta = 1/(1 + \tan^2 \theta)$ to express Ω_{\pm} in terms of ω_a, ω_b and g .

3) The ground state force* (3 Points). Consider the ground state energy of the coupled quantum harmonic oscillators in question 2) above. Assume that the coupling strength depends on the distance L between the oscillators according to $g(L) = \alpha L e^{-L/\lambda}$, where α and λ are two constants. Make a plot of the force acting between the two oscillators.

— 13 Points —

— To be handed in prior to the tutorials on 21.04.2017 —