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THEORETICAL OPTICS & PHOTONICS
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Discrete Quantum Optics

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1. **Driven Harmonic Oscillator:** The hamiltonian for a driven harmonic oscillator is given by

$$H(t) = \omega \hat{a}^\dagger \hat{a} + f(t)\left(\hat{a}^\dagger + \hat{a}\right), \tag{1}$$

where the function $f(t)$ represents an external driving force. In order to solve the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = H(t)\left|\Psi(t)\right\rangle, \tag{2}$$

we propose the transformation

$$\left|\Psi(t)\right\rangle = \exp\left(-\mathrm{i}\omega t \hat{a}^\dagger \hat{a}\right)\left|\Phi(t)\right\rangle, \tag{3}$$

note that $|\Psi(0)\rangle = |\Phi(0)\rangle$. Substituting $|\Psi(t)\rangle$ into Eq. (2) we obtain a differential equation for $|\Phi(t)\rangle$

$$\begin{aligned}\mathrm{i}\frac{\partial}{\partial t}\left|\Phi(t)\right\rangle = {} & f(t)\exp\left(\mathrm{i}\omega t\hat{a}^\dagger\hat{a}\right)\hat{a}^\dagger\exp\left(-\mathrm{i}\omega t\hat{a}^\dagger\hat{a}\right)\left|\Phi(t)\right\rangle \\ & + f(t)\exp\left(\mathrm{i}\omega t\hat{a}^\dagger\hat{a}\right)\hat{a}\exp\left(-\mathrm{i}\omega t\hat{a}^\dagger\hat{a}\right)\left|\Phi(t)\right\rangle.\end{aligned} \tag{4}$$

Using the identities

$$\begin{aligned}\exp\left(\mathrm{i}\omega t\hat{a}^\dagger\hat{a}\right)\hat{a}\exp\left(-\mathrm{i}\omega t\hat{a}^\dagger\hat{a}\right) &= \hat{a}\exp(-\mathrm{i}\omega t), \\ \exp\left(\mathrm{i}\omega t\hat{a}^\dagger\hat{a}\right)\hat{a}^\dagger\exp\left(-\mathrm{i}\omega t\hat{a}^\dagger\hat{a}\right) &= \hat{a}^\dagger\exp(\mathrm{i}\omega t),\end{aligned} \tag{5}$$

Eq. (4) reduces to

$$\mathrm{i}\frac{\partial}{\partial t}\left|\Phi(t)\right\rangle = \left(f(t)\exp\left(\mathrm{i}\omega t\right)\hat{a}^\dagger + f(t)\exp\left(-\mathrm{i}\omega t\right)\hat{a}\right)\left|\Phi(t)\right\rangle. \tag{6}$$

In order to solve Eq. (6) we use the ansatz

$$\left|\Phi(t)\right\rangle = \exp\left(A(t)\right)\exp\left(B(t)\hat{a}^\dagger\right)\exp\left(C(t)\hat{a}\right)\left|\Phi(0)\right\rangle. \tag{7}$$

Taking the time derivative of $|\Phi(t)\rangle$ yields

$$\begin{aligned}\mathrm{i}\frac{\partial}{\partial t}\left|\Phi(t)\right\rangle = {} & \mathrm{i}\frac{\partial A(t)}{\partial t}\left|\Phi(t)\right\rangle + \mathrm{i}\frac{\partial B(t)}{\partial t}\hat{a}^{\dagger}\left|\Phi(t)\right\rangle \\ & + \mathrm{i}\frac{\partial C(t)}{\partial t}\exp\left(A(t)\right)\exp\left(B(t)\hat{a}^{\dagger}\right)\hat{a}\hat{1}\exp\left(C(t)\hat{a}\right)\left|\Phi(0)\right\rangle,\end{aligned} \tag{8}$$

where we have tacitly introduced the operator $\hat{1} = \exp\left(-B(t)\hat{a}^{\dagger}\right)\exp\left(B(t)\hat{a}^{\dagger}\right)$. Then by using the identity $\exp\left(B(t)\hat{a}^{\dagger}\right)\hat{a}\exp\left(-B(t)\hat{a}^{\dagger}\right) = \hat{a} - B(t)$, Eq. (8) reduces to

$$\begin{aligned}\mathrm{i}\frac{\partial}{\partial t}\left|\Phi(t)\right\rangle = {} & \mathrm{i}\frac{\partial A(t)}{\partial t}\left|\Phi(t)\right\rangle + \mathrm{i}\frac{\partial B(t)}{\partial t}\hat{a}^{\dagger}\left|\Phi(t)\right\rangle \\ & + \mathrm{i}\frac{\partial C(t)}{\partial t}\left(\hat{a} - B(t)\right)\left|\Phi(t)\right\rangle.\end{aligned} \tag{9}$$

Since $\left|\Phi(t)\right\rangle$ must satisfy equation (6), the terms on the right hand side of Eq. (9) have to be equal to the terms on the right hand side of Eq. (6)

$$\begin{aligned}& \mathrm{i}\frac{\partial A(t)}{\partial t}\left|\Phi(t)\right\rangle + \mathrm{i}\frac{\partial B(t)}{\partial t}\hat{a}^{\dagger}\left|\Phi(t)\right\rangle + \mathrm{i}\frac{\partial C(t)}{\partial t}\left(\hat{a} - B(t)\right)\left|\Phi(t)\right\rangle \\ & \quad = \left(f(t)\exp\left(\mathrm{i}\omega t\right)\hat{a}^{\dagger} + f(t)\exp\left(-\mathrm{i}\omega t\right)\hat{a}\right)\left|\Phi(t)\right\rangle.\end{aligned} \tag{10}$$

From this equation we identify the differential equations for $A(t)$, $B(t)$, and $C(t)$

$$\begin{aligned}\frac{\partial A(t)}{\partial t} &= B(t)\frac{\partial C(t)}{\partial t}, \\ \frac{\partial B(t)}{\partial t} &= -\mathrm{i}f(t)\exp\left(\mathrm{i}\omega t\right), \\ \frac{\partial C(t)}{\partial t} &= -\mathrm{i}f(t)\exp\left(-\mathrm{i}\omega t\right).\end{aligned} \tag{11}$$

And the solutions are

$$\begin{aligned}A(t) &= -\int_0^t f(t')\exp\left(-\mathrm{i}\omega t'\right)\left(\int_0^{t'} f(t'')\exp\left(\mathrm{i}\omega t''\right)\mathrm{d}t''\right)\mathrm{d}t', \\ B(t) &= -\mathrm{i}\int_0^t f(t')\exp\left(\mathrm{i}\omega t'\right)\mathrm{d}t', \\ C(t) &= -\mathrm{i}\int_0^t f(t')\exp\left(-\mathrm{i}\omega t'\right)\mathrm{d}t'.\end{aligned} \tag{12}$$

Once we know the functions $A(t)$, $B(t)$, and $C(t)$ for a given driving force $f(t)$ we can compute the solution (7). In turn, from Eq. (3) and using the fact that $\left|\Psi(0)\right\rangle = \left|\Phi(0)\right\rangle$, we can compute the complete solution

$$\left|\Psi(t)\right\rangle = \exp\left(A(t)\right)\exp\left(-\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}\right)\exp\left(B(t)\hat{a}^{\dagger}\right)\hat{1}\exp\left(C(t)\hat{a}\right)\left|\Psi(0)\right\rangle, \tag{13}$$

where again we have introduced the identity operator $\hat{1} = \exp\left(\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}\right)\exp\left(-\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}\right)$. Then using identity

$$\begin{aligned}\exp\left(-\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}\right)\exp\left(B(t)\hat{a}^{\dagger}\right)\exp\left(\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}\right) &= \exp\left(B(t)\exp\left(-\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}\right)\hat{a}^{\dagger}\exp\left(\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}\right)\right) \\ &= \exp\left(B(t)\hat{a}^{\dagger}\exp\left(-\mathrm{i}\omega t\right)\right).\end{aligned} \tag{14}$$

Eq. (13) becomes

$$|\Psi(t)\rangle = \exp\left(A(t)\right)\exp\left(B(t)\exp\left(-\mathrm{i}\omega t\right)\hat{a}^{\dagger}\right)\exp\left(-\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}\right)\exp\left(C(t)\hat{a}\right)|\Psi(0)\rangle\,. \tag{15}$$

Assuming that at $t = 0$ the oscillator is prepared in the number state $|k\rangle$, we can compute the transition probability amplitudes

$$C_m(t) = \exp\left(A(t)\right)\langle m|\exp\left(B(t)\exp\left(-\mathrm{i}\omega t\right)\hat{a}^{\dagger}\right)\exp\left(-\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}\right)\exp\left(C(t)\hat{a}\right)|k\rangle\,. \tag{16}$$

For that we need the Taylor expansions

$$\exp\left(C(t)\hat{a}^{\dagger}\right)|k\rangle = \sum_{l=0}^{\infty}\frac{(C(t))^l}{l!}(\hat{a}^{\dagger})^l|k\rangle = \sum_{l=0}^{k}\frac{(C(t))^l}{l!}\sqrt{\frac{k!}{(k-l)!}}\,|k-l\rangle\,, \tag{17}$$

and

$$\begin{aligned}\langle m|\exp\left(B(t)\exp\left(-\mathrm{i}\omega t\right)\hat{a}^{\dagger}\right) &= \sum_{n=0}^{\infty}\frac{(B(t)\exp\left(-\mathrm{i}\omega t\right))^n}{n!}\langle m|\,\hat{a}^n \\ &= \sum_{n=0}^{m}\frac{(B(t)\exp\left(-\mathrm{i}\omega t\right))^n}{n!}\sqrt{\frac{m!}{(m-n)!}}\,\langle m-n|\,.\end{aligned} \tag{18}$$

Therefore, we have the expression

$$\sum_{l=0}^{k}\sum_{n=0}^{m}\frac{(B(t)\exp\left(-\mathrm{i}\omega t\right))^n}{n!}\frac{(C(t))^l}{l!}\sqrt{\frac{m!\quad k!}{(m-n)!(k-l)!}}\exp\left(-\mathrm{i}\omega t(k-l)\right)\langle m-n|k-l\rangle\,. \tag{19}$$

From the orthonormality condition, $\langle m-n|k-l\rangle = \delta_{m-n,k-l}$, we have two cases: $n = m+l-k$ and $l = k+n-m$. This allows us to get rid of the sum in n and l, respectively. By doing so, assuming $m = k+s$ and $m = k-s$, then multiplying by $\sqrt{\frac{(k+s)!}{(k+s)!}}$ and $\sqrt{\frac{k!}{k!}}$ in the first and second case, respectively, and after some algebra we obtain a closed form expression for Eq. (19)

$$C_m(t) = \exp\left(A(t) - \mathrm{i}\omega m t\right)\times\begin{cases}(-B^*(t))^{k-m}\sqrt{\dfrac{m!}{k!}}L_m^{k-m}(\mid B\mid^2) & \text{for}\qquad m \le k. \\ (B(t))^{m-k}\sqrt{\dfrac{k!}{m!}}L_k^{m-k}(\mid B\mid^2) & \text{for}\qquad m \ge k.\end{cases} \tag{20}$$

On the other hand we expand the state vector $|\Psi(t)\rangle$ in terms of number states

$$|\Psi(t)\rangle = \sum_{m=0}^{\infty}|m\rangle\langle m|\Psi(t)\rangle = \sum_{m=0}^{\infty}C_m(t)\,|m\rangle\,, \tag{21}$$

where we have defined $C_m(t) = \langle m|\Psi(t)\rangle = \langle m|\,U(t)\,|k\rangle$. Substituting this expansion into the Schrödinger equation yields

$$\mathrm{i}\frac{\partial C_m(t)}{\partial t} = \omega m C_m(t) + f(t)\left(\sqrt{m}C_{m-1}(t) + \sqrt{m+1}C_{m+1}(t)\right). \tag{22}$$

Or in matrix form

$$\mathrm{i}\frac{\partial}{\partial t}\begin{bmatrix} C_0(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & f(t)\sqrt{1} & 0 & \dots & 0 \\ f(t)\sqrt{1} & \omega & f(t)\sqrt{2} & \dots & 0 \\ 0 & f(t)\sqrt{2} & 2\omega & \dots & \vdots \\ 0 & 0 & f(t)\sqrt{3} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 \end{bmatrix}\begin{bmatrix} C_0(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \\ \vdots \end{bmatrix}.$$

The next step is to write a Matlab script to solve this system of coupled differential equations using the Runge-Kutta method. Consider $f(t) = \kappa_0 + \epsilon\cos(\omega_1 t)$ with $\kappa_0 = 1$, $\omega = 1$, and analyse the cases when $\omega_1 = \frac{3}{4}, \frac{2}{3}$. Additionally, describe your observations in the resonant case $\omega = \omega_1 = 1$. For your simulations, consider a number of energy levels $m = 0, ..., 50$, and the initial states $|k\rangle = |0\rangle, |2\rangle, |5\rangle$, compare the analytical solution (20) with your numerical results. Plot the function $\mid B \mid^2$ on top of your simulations, what do you see?