

## Discrete Quantum Optics

Dr. Armando Perez-Leija and Prof. Dr. Kurt Busch  
M.Sc. Paraschos Varytis

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1. Consider a quantum system, initially prepared in a number state  $|\Psi(0)\rangle = |k\rangle$ , governed by the hamiltonian  $H = (\hat{a}^\dagger + \hat{a})$ .

In order to solve the Schrödinger equation

$$i \frac{\partial}{\partial t} |\Psi(t)\rangle = H |\Psi(t)\rangle, \quad (1)$$

we start by expanding the state vector  $|\Psi(t)\rangle$  in terms of number states

$$|\Psi(t)\rangle = \sum_{m=0}^{\infty} |m\rangle \langle m|\Psi(t)\rangle = \sum_{m=0}^{\infty} C_m(t) |m\rangle, \quad (2)$$

where we have defined  $C_m(t) = \langle m|\Psi(t)\rangle = \langle m|U(t)|k\rangle$ . Substituting expansion (2) into the Schrödinger equation yields

$$i \sum_{m=0}^{\infty} \frac{\partial C_m(t)}{\partial t} |m\rangle = \sum_{m=0}^{\infty} C_m(t) \hat{a}^\dagger |m\rangle + \sum_{m=0}^{\infty} C_m(t) \hat{a} |m\rangle. \quad (3)$$

Since  $\hat{a}^\dagger |m\rangle = \sqrt{m+1} |m+1\rangle$  and  $\hat{a} |m\rangle = \sqrt{m} |m-1\rangle$  we have

$$i \sum_{m=0}^{\infty} \frac{\partial C_m(t)}{\partial t} |m\rangle = \sum_{m=0}^{\infty} C_m(t) \sqrt{m+1} |m+1\rangle + \sum_{m=0}^{\infty} C_m(t) \sqrt{m} |m-1\rangle. \quad (4)$$

Multiplying to the left by  $\langle n|$  we obtain

$$i \sum_{n=0}^{\infty} \frac{\partial C_m(t)}{\partial t} \langle n|m\rangle = \sum_{m=0}^{\infty} C_m(t) \sqrt{m+1} \langle n|m+1\rangle + \sum_{m=0}^{\infty} C_m(t) \sqrt{m} \langle n|m-1\rangle. \quad (5)$$

Using the orthogonality relation  $\langle n|m\rangle = \delta_{n,m}$ , Eq. (5) reduces to

$$i \frac{\partial C_m(t)}{\partial t} = \sqrt{m} C_{m-1}(t) + \sqrt{m+1} C_{m+1}(t). \quad (6)$$

Or in matrix form

$$i \frac{\partial}{\partial t} \begin{bmatrix} C_0(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{1} & 0 & \dots & 0 \\ \sqrt{1} & 0 & \sqrt{2} & \dots & 0 \\ 0 & \sqrt{2} & 0 & \dots & \vdots \\ 0 & 0 & \sqrt{3} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 \end{bmatrix} \begin{bmatrix} C_0(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \\ \vdots \end{bmatrix}.$$

Instead of solving the set of coupled differential equation (6), it is easier to compute  $C_m(t)$  directly from the formal solution of the Schrödinger equation

$$|\Psi(t)\rangle = \exp(-it(\hat{a}^\dagger + \hat{a})) |k\rangle. \quad (7)$$

Since  $[\hat{a}, \hat{a}^\dagger] = 1$ , we can use the Baker-Campbell-Hausdorff formula to split the exponential operator

$$\exp(-it(\hat{a}^\dagger + \hat{a})) = \exp\left(-\frac{t^2}{2}\right) \exp(-it\hat{a}^\dagger) \exp(-it\hat{a}). \quad (8)$$

Therefore, we have

$$|\Psi(t)\rangle = \exp\left(-\frac{t^2}{2}\right) \exp(-it\hat{a}^\dagger) \exp(-it\hat{a}) |k\rangle. \quad (9)$$

Since  $C_m(t) = \langle m|\Psi(t)\rangle = \langle m|U(t)|k\rangle$ , we have to multiply Eq. (9) to the left by  $\langle m|$

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right) \langle m|\exp(-it\hat{a}^\dagger) \exp(-it\hat{a}) |k\rangle. \quad (10)$$

We now use the Taylor expansions

$$\exp(-it\hat{a}^\dagger) |k\rangle = \sum_{l=0}^{\infty} \frac{(-it)^l}{l!} (\hat{a}^\dagger)^l |k\rangle = \sum_{l=0}^k \frac{(-it)^l}{l!} \sqrt{\frac{k!}{(k-l)!}} |k-l\rangle, \quad (11)$$

and

$$\langle m|\exp(-it\hat{a}) = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \langle m|\hat{a}^n = \sum_{n=0}^m \frac{(-it)^n}{n!} \sqrt{\frac{m!}{(m-n)!}} \langle m-n|, \quad (12)$$

to write Eq. (10) as

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right) \sum_{n=0}^m \sum_{l=0}^k \frac{(-it)^n (-it)^l}{n! l!} \sqrt{\frac{m! k!}{(m-n)!(k-l)!}} \langle m-n|k-l\rangle. \quad (13)$$

Notice that the orthonormality condition,  $\langle m-n|k-l\rangle = \delta_{m-n, k-l}$ , allows us to take  $n = m + l - k$ , and as a result Eq. (13) reduces to

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right) \sum_{l=0}^k \frac{(-it)^{2l} (-it)^{m-k}}{(m+l-k)! l!} \sqrt{\frac{m! k!}{(m-n)!(k-l)!}}. \quad (14)$$

Moreover, taking  $m = k + s$  and multiplying by  $\sqrt{\frac{(k+s)!}{(k+s)!}}$  we obtain

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right) (-it)^s \sqrt{\frac{k!}{(k+s)!}} \sum_{l=0}^k \frac{(-1)^l (t^2)^l (k+s)!}{l!(l+s)!(k-l)!}. \quad (15)$$

From this expression we identify the associated Laguerre polynomials

$$L_k^s(t^2) = \sum_{l=0}^k \frac{(-1)^l (t^2)^l (k+s)!}{l!(l+s)!(k-l)!}. \quad (16)$$

And Eq. (15) becomes

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right) (-it)^s \sqrt{\frac{k!}{(k+s)!}} L_k^s(t^2) \quad \text{for} \quad m = k + s. \quad (17)$$

Since this expression is valid for  $m = k + s$ , we can take  $s = m - k$  to obtain

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right) (-it)^{m-k} \sqrt{\frac{k!}{m!}} L_k^{m-k}(t^2) \quad \text{for} \quad m \geq k. \quad (18)$$

Starting again from Eq. (13), and using the orthonormality condition  $\langle m-n | k-l \rangle = \delta_{m-n, k-l}$  we now take  $l = n + k - m$  to get rid of the sum in  $l$ . Then, taking  $m = k - s$  and multiplying by  $\sqrt{\frac{k!}{k!}}$  we obtain

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right) (-it)^s \sqrt{\frac{(k-s)!}{k!}} L_{k-s}^s(t^2) \quad \text{for} \quad m = k - s. \quad (19)$$

Finally, taking  $s = k - m$  yields

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right) (-it)^{k-m} \sqrt{\frac{m!}{k!}} L_m^{k-m}(t^2) \quad \text{for} \quad m \leq k. \quad (20)$$

Collecting both parts of the solution we have the analytical expression for the probability amplitudes  $C_m(t)$

$$C_m(t) = \exp\left(-\frac{t^2}{2}\right) \times \begin{cases} (-it)^{k-m} \sqrt{\frac{m!}{k!}} L_m^{k-m}(t^2) & \text{for} \quad m \leq k. \\ (-it)^{m-k} \sqrt{\frac{k!}{m!}} L_k^{m-k}(t^2) & \text{for} \quad m \geq k. \end{cases} \quad (21)$$

This expressions describe how the different states  $|m\rangle$  will be populated when the system starts in a number state  $|k\rangle$ . The next step is to write a Matlab script to solve the system of coupled differential equations (6) using the Runge-Kutta method. In doing so consider  $m = 0, \dots, 50$  and the initial states  $|k\rangle = |0\rangle, |2\rangle, |5\rangle$ , compare the analytical and numerical solutions.