Humboldt–Universität zu Berlin
Institut für Physik

SoSe 2017

Online on the 09.05.2017

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Fluctuation-induced Phenomena: Problem Set 3

1) Calculation of the Casimir effect in one dimension. (7 Points)
Consider two perfectly conducting plates separated by a distance L, as illustrated in Fig. 1.

Figure 1: Geometry for the one dimensional Casimir calculation: two perfectly conducting plates separated by a distance L.

1. First argue that the zero-point energy of the e.m. field between the plates is given as

$$\mathcal{E}_{\text{disc}} = \frac{\pi \hbar c}{L} \sum_{n=1}^{\infty} n. \tag{1}$$

Hint: Find the entire spectrum of modes by solving the wave equation with suitable boundary conditions. Then use the postulate that each mode carries a zero point contribution to the energy of $\hbar\omega/2$.

2. Next, we consider the case where there are no mirrors (or, equivalently, the limit where L becomes large). In this limit, show that the zero-point energy in the same length L can be written as

$$\mathcal{E}_{\text{cont}} = \frac{\pi \hbar c}{L} \int_0^{\infty} n \mathrm{d}n. \tag{2}$$

3. The change in the zero-point energy is evidently

$$\Delta\mathcal{E} = \frac{\pi \hbar c}{L} \left\{ \sum_{n=1}^{\infty} n - \int_0^{\infty} n \mathrm{d}n \right\}. \tag{3}$$

Following Casimir, we can evaluate eq. (3) by use of the Euler-Maclaurin formula

$$\sum_{n=a}^{b} f_n - \int_a^b f(n)\mathrm{d}n = -\frac{1}{2}\left(f(b) + f(a)\right) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right), \tag{4}$$

where B_k are the Bernoulli numbers. To this end, we can use a regularization of the integrand by noting that

$$\Delta\mathcal{E} = \frac{\pi\hbar c}{L}\lim_{\epsilon\to 0}\left\{\sum_{n=1}^{\infty} n e^{-\epsilon n} - \int_0^{\infty} n e^{-\epsilon n}\mathrm{d}n\right\}. \tag{5}$$

With Eqs. (4) and (5), calculate the Casimir force as minus the gradient of the (renormalized) energy $\Delta\Delta\mathcal{E}(L)$.

4. More than two centuries ago Niels Henrik Abel and Giovanni Antonio Amedeo Plana discovered independently the following summation formula

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty}\mathrm{d}x\, f(x) + \frac{f(0)}{2} + \mathrm{i}\int_0^{\infty}\mathrm{d}t\,\frac{f(\mathrm{i}t)-f(-\mathrm{i}t)}{e^{2\pi t}-1}. \tag{6}$$

Use Eq: (6) to calculate the Casimir force.

2) Casimir oscillator (7 Points)

Consider the set-up in Figure 2. The spring constant K is known, as is the mass M of each of the two metallic plates. At equilibrium, the distance between the two plates is d. The left plate is now displaced from the equilibrium by a small distance, causing the system to oscillate.

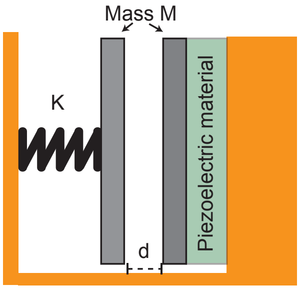

Figure 2: Set up of a Casimir oscillator consisting of two plates of area A and mass M separated by a distance d. One plate is attached to a spring with spring constant K, the other is attached to a piezoelectric material to provide precise control of the distance.

(a) Show that the Casimir force between the plates leads to an oscillation frequency which depends on the distance d (as well as the area of the plates). Calculate the frequency shift with respect to the oscillator's eigenfrequency.

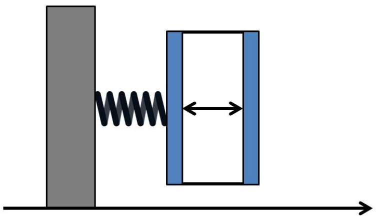

Figure 3: Set up of a Casimir oscillator the two plates of area A and mass M are connected (at the edges) and disconnected from the piezoelectric material. The cavity is still attached to a spring with spring constant K.

(b) As a thought experiment, suppose now that the whole cavity is oscillating (i.e. the two plates are connected (at the edges) and disconnected from the piezoelectric material, see Fig. 3). Use the mass-energy equivalence to show that also in this case the frequency is expected to depend on the distance between the plates.

— 14 Points —

— To be handed in prior to the tutorials on 15.05.2017 —