

Fluctuation-induced Phenomena: Problem Set 9

- 1) **Free space Green tensor** (5 Points). In free space, the Green tensor $\mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega)$ is the solution to the equation (frequency space formulation)

$$\nabla \times \nabla \times \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) - k_0^2 \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}'), \quad (1)$$

where $k_0 = \omega/c$ is the ratio of the angular frequency to the speed of light.

- (a) Take the divergence of both sides of Eq. (1) and show that

$$\nabla \cdot \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = -\frac{1}{k_0^2} \nabla \delta(\mathbf{r} - \mathbf{r}'). \quad (2)$$

- (b) Use the vectorial identity $\nabla \times \nabla \times \mathbf{A} = \nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}$ to rewrite Eq. (1) as

$$\nabla^2 \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) + k_0^2 \mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = - \left[1 + \frac{1}{k_0^2} \nabla \nabla \right] \delta(\mathbf{r} - \mathbf{r}'). \quad (3)$$

- (c) If we now define

$$\mathbf{G}_0(\mathbf{r}, \mathbf{r}', \omega) = - \left[1 + \frac{1}{k_0^2} \nabla \nabla \right] g_0(\mathbf{r}, \mathbf{r}', \omega), \quad (4)$$

show that $g_0(\mathbf{r}, \mathbf{r}', \omega)$ is the (scalar) Green function of the Helmholtz equation:

$$\nabla^2 g_0(\mathbf{r}, \mathbf{r}', \omega) + k_0^2 g_0(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r}, \mathbf{r}', \omega). \quad (5)$$

- 2) **The Green function of the Helmholtz equation** (5 Points).

- (a) By transforming to the spatial Fourier space, show that the Green tensor of the Helmholtz equation is given by the integral

$$g_0(\mathbf{r}, \mathbf{r}', \omega) = -\frac{1}{(2\pi)^3} \int \frac{e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}}{|\mathbf{k}|^2 - k_0^2} d^3\mathbf{k}. \quad (6)$$

- (b) Calculate the integral considering the boundary condition that for a fixed \mathbf{r}' , $g_0(\mathbf{r}, \mathbf{r}', \omega)$ must be an outgoing wave (traveling away from \mathbf{r}') in the limit $|\mathbf{r} - \mathbf{r}'| \rightarrow \infty$. This condition is known as the Sommerfeld radiation condition.

Hint: You can control the boundary condition by a small displacement of the pole of the integrand.

— 10 Points —

— To be handed in prior to the tutorials on 26.06.2018 —