Discrete Quantum Optics

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1. If n and m are integers and $\hat{a}$ and $\hat{a}^\dagger$ are the boson annihilation and creation operators, respectively, show that

- $\exp\left(\mathrm{i}t\hat{a}\right)|k\rangle = \sum_{l=0}^{k} \frac{(\mathrm{i}t)^l}{l!}\sqrt{\frac{k!}{(k-l)!}}\,|k-l\rangle \quad for \quad l \leq k.$
- $\langle m|\exp\left(\mathrm{i}t\hat{a}^\dagger\right) = \sum_{n=0}^{m} \frac{(\mathrm{i}t)^n}{n!}\sqrt{\frac{m!}{(m-n)!}}\,\langle m-n| \quad for \quad n \leq m.$
- $\left[\hat{a}^\dagger\hat{a}, \hat{a}^{\dagger m}\right] = m\hat{a}^{\dagger m}$
- $\left[\hat{a}^\dagger\hat{a}, \hat{a}^{m}\right] = -m\hat{a}^{m}$
- $\exp\left(\xi\hat{a}\right) f(\hat{a}^\dagger)\,|0\rangle = f(\hat{a}^\dagger + \xi)\,|0\rangle$
- $\exp\left(\xi\hat{a}^\dagger\hat{a}\right) f(\hat{a}^\dagger)\,|0\rangle = f\left(\hat{a}^\dagger\exp(-\xi)\right)|0\rangle$
- $\exp\left(\xi\hat{a}^\dagger\right)\hat{a}^m = (\hat{a}-\xi)^m \exp\left(\xi\hat{a}^\dagger\right)$
- $\exp\left(\xi\hat{a}\right)\hat{a}^{\dagger m} = (\hat{a}^\dagger+\xi)^m \exp\left(\xi\hat{a}\right)$

2. Two coupled oscillators are described by the hamiltonian

$$H = \omega_1\hat{a}^\dagger\hat{a} + \omega_1\hat{b}^\dagger\hat{b} + \kappa\left(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger\right).$$

- Consider $\omega_1 = \omega_2 = \omega$ and solve the Heisenberg equations of motion for $\hat{a}^\dagger(t)$ and $\hat{b}^\dagger(t)$.
- Compute the expressions $\hat{a}^\dagger_H(t) = U^\dagger(t)\hat{a}^\dagger_S U(t)$ and $\hat{b}^\dagger_H(t) = U^\dagger(t)\hat{b}^\dagger_S U(t)$, where the subscripts H and S designate the Heisenberg and the Schrödinger pictures, respectively. Compare your results with the solutions of the corresponding Heisenberg equations.
- Think about the case when $\omega_1 \neq \omega_2$, compute the eigenvectors and eigenvalues for the hamiltonian and use the spectral decomposition to find the evolution operator.

- Write a Matlab script to compute the solution using the Runge-Kutta method. Compare the analytical and numerical results.

3. The evolution of a quantum particle with mass $m = 1$ in free space is governed by the Schrödinger equation

$\mathrm{i}\frac{\partial}{\partial t}\left|\Psi(x,t)\right\rangle = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\left|\Psi(x,t)\right\rangle.$

Thus, by defining the momentum operator $\hat{p} = -\mathrm{i}\frac{\partial}{\partial x}$, such that the commutator $[\hat{x},\hat{p}] = \mathrm{i}$, it is clear that the formal solution is

$\left|\Psi(x,t)\right\rangle = \exp\left(\frac{\mathrm{i}t\hat{p}^2}{2}\right)\left|\Psi(x,0)\right\rangle.$

Assume that the initial wavefunction is given as the product of two functions $\left|\Psi(x,0)\right\rangle = g(x)f(x)$. Develop the expression for $\left|\Psi(x,t)\right\rangle$ utilizing the identities given in exercise number 10 in the first set of exercises. Start by expanding the initial wavefunction $\left|\Psi(x,0)\right\rangle = g(x)f(s)$ in plane waves, namely

$\left|\Psi(x,t)\right\rangle = \exp\left(\frac{\mathrm{i}t\hat{p}^2}{2}\right)\left[\int_{-\infty}^{\infty} G(u)\exp\left(\mathrm{i}xu\right)du\right]\left[\int_{-\infty}^{\infty} F(v)\exp\left(\mathrm{i}xv\right)dv\right].$

Write your result for the particular case when $g(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right)$. In practice this Gaussian function is known as apodization function and it is used to create finite wavefunctions.